# Accurate Approximations for a Function Appearing in the Analysis of Microstrip

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***Abstract*—The function $Q(x)$ defined in the body of the paper is approximated accurately over the entire range $-1 \leqslant x < +1$. The simple approximate formulas are then used to obtain some simple expressions for several quantities which are encountered in the quasistatic analysis of microstrip transmission line.**

## I. Introduction

The aim of this short paper is to obtain simple, closed-form approximations to the function

$$Q(x)=\sum_{m=1}^{\infty} x^m \ln\left(\frac{m+1}{m}\right) \tag{1}$$

which will be accurate over the range of $-1 \leqslant x < +1 (Q(+1)=\infty)$. The function $Q$ has appeared in many analyses of the quasistatic properties of open microstrip transmission lines [1]–[8], either in the form of (1), or in the guise of a related function $Q_0(x)$

$$Q_0(x)=\sum_{m=1}^{\infty} x^m \ln m = \left(\frac{x}{1-x}\right) Q(x). \tag{2}$$

The argument $x$ is of the form $-\delta_\epsilon = (1-\epsilon_r)/(1+\epsilon_r)$ when $Q$ figures into the capacitance of wide or narrow microstrip, or of the form $\delta_\mu = (\mu_r - 1)/(\mu_r + 1)$ in expressions for the line inductance, where $\epsilon_r$ and $\mu_r$ are, respectively, the relative permittivity and relative permeability of the microstrip substrate. The expressions $Q(-\delta_\epsilon)$ and $Q(\delta_\mu)$ both figure into the expression for the quasistatic edge admittance of a conducting half-plane on a grounded substrate [4], [9].

While $Q$ can be evaluated from its series form (1), it is cumbersome to do so numerically, especially for large $\epsilon_r$ or $\mu_r$. It is also possible to relate $Q$ to a class of generalized zeta-functions [9], but this does not appear to be of help in its numerical calculation either. If $Q$ could be approximated in closed form using only elementary functions, the expressions for line capacitance, inductance, and edge admittance [1]–[9] for microstrip would become quite simple, and make them more attractive for design work on small calculators.

In point of fact, Wheeler [10] several years ago provided a unified formula for the characteristic impedance of microstrip. This formula was said to be the result of matching the correct limiting behavior for very narrow and very wide microstrips with a sufficiently simple functional form, and it turns out to be accurate to within 2 percent for all values of the microstrip parameters in the case of a nonmagnetic substrate ($\mu_r = 1$). However, nowhere in Wheeler's expression does the function $Q$ appear. If we compare the narrow-strip limit of Wheeler's formula with the results of [1]–[3], [5], we find that they would be identical if

$$Q(-\delta_\epsilon) \to \ln\left(\frac{7\epsilon_r + 4}{11\epsilon_r}\right) \tag{3}$$

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TABLE I
COMPARISON OF EXACT AND APPROXIMATE VALUES FOR $Q(x)$

| x | exact | eqn. (4) | eqns.(5)-(6) |
|---|---|---|---|
| -1.0 | -0.4516 | -0.4520 | -0.4516 |
| -0.7 | -0.3509 | -0.3559 | -0.3478 |
| -0.3 | -0.1778 | -0.1837 | -0.1741 |
| 0 | 0 | 0 | 0 |
| 0.3 | 0.2546 | 0.2713 | 0.2459 |
| 0.7 | 0.9146 | 0.9921 | 0.8840 |
| 0.99 | 4.0524 | 4.2905 | 4.0391 |

i.e., if

$$Q(x) \to \ln\left(1+\frac{8}{11}\frac{x}{1-x}\right). \tag{4}$$

Some numerical checking shows that, indeed, the replacement (4) is quite accurate (to 5 percent near $x=0$ and improving to about 0.1 percent near $x=-1$) over the range $-1 \leqslant x \leqslant 0$ covered by $-\delta_\epsilon$. In the range $0 \leqslant x < 1$, however, which is covered by $\delta_\mu$, the approximation is far less accurate, especially for $x$ close to 1 (corresponding to highly magnetic substrates). In an effort to improve the approximation over this range, we are led to consider an approximation of the form

$$Q(x) \simeq \frac{1}{2}\ln\left[1+A\frac{x}{1-x}+B\left(\frac{x}{1-x}\right)^2\right], \qquad |x|<1 \tag{5}$$

where $A$ and $B$ are suitably chosen constants ((4) is a special case of (5), with $A=16/11 \simeq 1.4545$ and $B=64/121 \simeq 0.5289$). If desired, we could motivate the choice of functions in (5) as an approximation to $dQ/du$ by a rational function of $u=x/(1-x)$.

With two constants $A$ and $B$ to adjust, we can achieve a better match to the function $Q$, using the following as guidelines:

1) $Q(0)=0$
2) $Q'(0)=\ln 2$
3) $Q(-1)=\ln(2/\pi)$
4) $Q(x) \to -\ln(1-x)-\gamma, \quad$ as $x \to +1$

where $\gamma = 0.5772\cdots$ is Euler's constant. Conditions 3) and 4) follow from the expression of $Q$ in terms of Lerch's zeta-function [1], [11]. Although the final choice of $A$ and $B$ is, to some extent, arbitrary, we find that

$$\left.\begin{aligned} A&=1.3471 \\ B&=0.3152 \end{aligned}\right\} \tag{6}$$

(so that 3) and 4) are met precisely, and 2) is satisfied to an excellent approximation) provide an overall agreement in (5), which is very good—less than 3 percent at its worst (near $x=0$). Note that this largest relative error in $Q$ occurs when $Q(x)$ is near zero, and will thus represent a much smaller relative error in a formula where $Q$ appears. A comparison of the accuracy of (5)–(6) versus that of (4) is shown in Table I.

## II. Applications

From (5), we have

$$Q(-\delta_\epsilon) \simeq \frac{1}{2}\ln r_\epsilon \tag{7}$$

$$Q(\delta_\mu) \simeq \frac{1}{2}\ln r_\mu \tag{8}$$

where

$$r_\epsilon = 0.4052 + \frac{0.5160}{\epsilon_r} + \frac{0.0788}{\epsilon_r^2} \tag{9}$$

and

$$r_\mu = 0.4052 + 0.516\mu_r + 0.0788\mu_r^2. \tag{10}$$

For a microstrip transmission line of width $2l$ on a magnetodielectric substrate of thickness $t$, relative permittivity $\epsilon_r$, and relative permeability $\mu_r$, we can now give the following formulas.

*A. Narrow Strips ($2l/t \ll 1$; cf. [1]–[3], [5])*

*1) Capacitance per unit length:*

$$C \simeq \frac{\pi\epsilon_0(\epsilon_r+1)}{\ln\left(\frac{4t\sqrt{r_\epsilon}}{l}\right)}. \tag{11}$$

*2) Inductance per unit length:*

$$L \simeq \frac{\mu_0\mu_r}{\pi(\mu_r+1)}\ln\left(\frac{4t\sqrt{r_\mu}}{l}\right). \tag{12}$$

*B. Wide Strips ($2l/t \gg 1$; cf. [1], [7], [8], and note an apparent error in the sign of $Q_0(-\delta_\epsilon)$ as it appears in [1])*

*1) Capacitance per unit length:*

$$C \simeq \epsilon_0\left\{\epsilon_r\frac{2l}{t} + \frac{2}{\pi}\left[1 + \ln\frac{l}{t\sqrt{r_\epsilon}} + \epsilon_r\ln\left(2\pi\sqrt{r_\epsilon}\right)\right]\right\}. \tag{13}$$

*2) Inductance per unit length:*

$$L \simeq \frac{\mu_0\mu_r}{(2l/t)+(2/\pi)\left[\mu_r\left(1+\ln\frac{l}{t\sqrt{r_u}}\right)+\ln\left(2\pi\sqrt{r_\mu}\right)\right]}. \tag{14}$$

Taking Wheeler's formula as a model, we can fashion the following expressions, which match the leading terms of (11)–(14) in both wide and narrow strip limits, and provide a smooth interpolation in between.

*C. Arbitrary Strip Width*

*1) Capacitance per unit length:*

$$C = \frac{2\pi\epsilon_0(\epsilon_r+1)}{\ln\left\{1+\frac{8t}{l}\left[\frac{tr_\epsilon}{l}+\sqrt{\left(\frac{tr_\epsilon}{l}\right)^2+\left(\frac{\pi(\epsilon_r+1)}{8\epsilon_r}\right)^2}\right]\right\}}. \tag{15}$$

*2) Inductance per unit length:*

$$L = \frac{\mu_0\mu_r}{2\pi(\mu_r+1)}\ln\left\{1+\frac{8t}{l}\left[\frac{tr_\mu}{l}+\sqrt{\left(\frac{tr_\mu}{l}\right)^2+\left(\frac{\pi(\mu_r+1)}{8}\right)^2}\right]\right\}. \tag{16}$$

Note that Wheeler's formula gives $Z_c = (L/C)^{1/2}$ directly, but not $L$ or $C$ individually, as do (15) and (16). Computation of $Z_c$ from (15) and (16) provides accuracy comparable to that of Wheeler's expression when $\mu_r = 1$, but is also capable of handling magnetic substrates, for which Wheeler's expression does not apply. The lack of truly accurate data for comparison when $\mu_r > 1$ makes the accuracy difficult to assess in this case.

The reflection coefficient of a TEM wave incident from underneath a substrate-loaded parallel-plate waveguide at an angle $\phi$ to the edge of the upper plate (Fig. 1) has the form [4], [9]

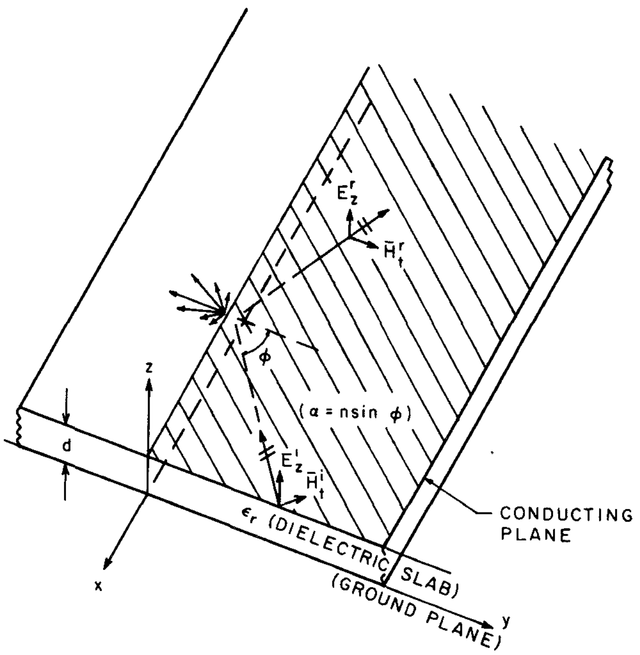


Fig. 1. TEM wave incident from underneath a substrate-loaded parallel-plate waveguide, $n = \sqrt{\mu_r\epsilon_r}$.

$$\Gamma_{TEM} = e^{i\chi(\sqrt{\mu_r\epsilon_r}\sin\phi)} \tag{17}$$

where, for an electrically thin ($\sqrt{\mu_r\epsilon_r}k_0 d \ll 1$) substrate, we can use (7) and (8) to obtain

$$\chi(\alpha) = \frac{2k_0 d}{\pi\sqrt{\mu_r\epsilon_r - \alpha^2}}\left\{(1-\alpha^2)\mu_r\left[\ln\left(k_0 d\sqrt{\alpha^2-1}\right)+\gamma-1\right]\right.$$
$$+ \mu_r\left[\ln\sqrt{r_\epsilon} - \alpha^2\ln\sqrt{r_\mu}\right]$$
$$\left. - \mu_r\epsilon_r\ln\left(2\pi\sqrt{r_\epsilon}\right) + \alpha^2\ln\left(2\pi\sqrt{r_\mu}\right)\right\} \tag{18}$$

where $d$ is the thickness of the substrate, and $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the free-space wavenumber. Under appropriate conventions, this expression could also be used to obtain a formula for the end admittance of this edge as well.

## III. Conclusion

We have given a simple, accurate, closed-form approximation for the function $Q(x)$ which appears in formulas for several properties of microstrip transmission lines. The approximation has been used to obtain simple expressions for a number of line parameters associated with microstrip structures.


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## References


[1] E. A. Shchapoval, "Capacitance, inductance and effective relative permittivity of microstrip line [Russian]," *Electron. Lett.*, vol. 11, pp. 225–226, 1975.

[2] A. T. Fialkovskii, "Dispersion in a nonsymmetrical strip line [Russian]," *Radiotekh. Elektron.*, vol. 21, pp. 441–450, 1976. (Engl. transl. in *Radio Eng. Electron. Phys.*, vol. 21, no. 3, pp. 7–15, 1976.)

[3] L. A. Vainshtein and A. T. Fialkovskii, "Modes in slot- and strip-line

waveguides; Variational method and simpler results [Russian]," *Radiotekh. Elektron.*, vol. 21, pp. 1137–1148, 1976. (Engl. trans. in *Radio Eng. Electron. Phys.*, vol. 21, no. 6, pp. 1–11, 1976.)

[4] E. I. Nefedov, N. P. Santalov, and A. T. Fialkovskii, "Dispersion characteristics of nonradiating waves in a microstrip waveguide [Russian]," *Radiotekh. Elektron.*, vol. 24, pp. 17–23, 1979. (Engl. transl. in *Radio Eng. Electron Phys.*, vol. 24, no. 1, pp. 15–20, 1979.)

[5] D. C. Chang and E. F. Kuester, "An analytic theory for narrow open microstrip," *Arch. Elek. Übertragung.*, vol. 33, pp. 199–206, 1979.

[6] E. F. Kuester and D. C. Chang, "Closed-form expressions for the current or charge distribution on parallel strips or microstrip," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 254–259, 1980; addendum, *ibid.*, p. 1143.

[7] S. Y. Poh, W. C. Chew, and J. A. Kong, "Approximate formulas for the line capacitance and characteristic impedance of microstrip line," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-29, pp. 135–142, 1981.

[8] W. C. Chew and J. A. Kong, "Asymptotic formula for the capacitance of two oppositely charged discs," *Math. Proc. Camb. Phil. Soc.*, vol. 89, pp. 373–384, 1981.

[9] E. F. Kuester, R. T. Johnk, and D. C. Chang, "The thin-substrate approximation for reflection from the end of a slab-loaded parallel-plate waveguide with application to microstrip patch antennas," *IEEE Trans. Antenna Propagat.*, vol. AP-30, pp. 910–917, 1982.

[10] H. A. Wheeler, "Transmission-line properties of a strip on a dielectric sheet on a plane," *IEEE Trans. Microwave Theory Tech.* vol. MTT-25, pp. 631–647, 1977.

[11] D. C. Chang and E. F. Kuester, "An analytic theory for narrow open microstrip," Electromagnetics Laboratory, Dept. of Elec Eng., Univ. of Colorado, Boulder, Sci. Rep. 28, May 1978.

# *W*-Band Low-Insertion-Loss *E*-Plane Filter

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*Abstract* —**Computer-optimized design data for a four-resonator metallic *E*-plane filter are given with a midband frequency of about 94 GHz. The method of field expansion into suitable eigenmodes used considers the effects of finite insert thickness and higher order mode interaction. The measured minimum passband insertion loss of a metal filter prototype is 1.5 dB.**

## I. Introduction

Increasing interest in the 3-mm region of millimeter waves has stirred the need for low-cost low-loss *W*-band filters. Pure metal inserts [1]–[4] placed in the *E*-plane of rectangular waveguides without any substrates (Fig. 1) achieve low-loss designs. Moreover, metal etching techniques [3], [4] can be applied for mass production. This paper yields computer-optimized design data for such a *W*-band filter. The computer-aided design algorithm is based on field expansion into eigenmodes [3]–[4] which, in contrast to the residue-calculus technique [5], includes the finite strip thickness considerably influencing passband ripple behavior and midband frequency, as has already been demonstrated in [3]. The measured frequency response of the metal insert filter that has been designed and operated at 94 GHz shows good agreement with theory.

## II. Theory and Design

As in [3], [4], the fields for each subregion ($\nu$) at the corresponding step discontinuities of the filter structure (Fig. 1) are derived from the $x$-component of the magnetic Hertzian vector

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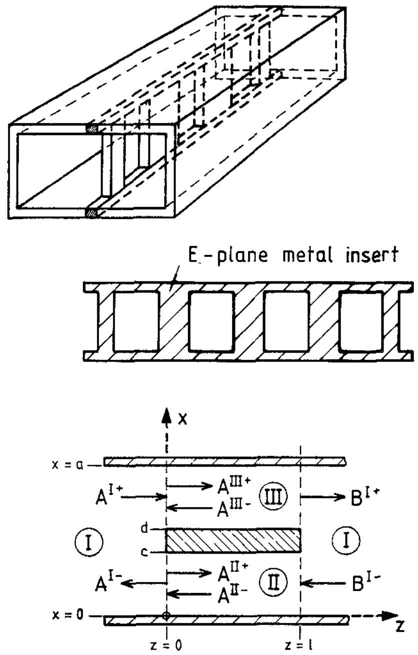


Fig. 1. Low-insertion-loss *E*-plane metal insert filter.

$\vec{\Pi}_h$, which is assumed to be a sum of suitable eigenmodes

$$\Pi_{hx}^{(\nu)} = \sum_{m=1}^{\infty} A^{(\nu)\pm} \cdot T_m^{(\nu)} \cdot \sin\left[\frac{m\pi}{p^{(\nu)}} \cdot f^{(\nu)}\right] e^{\mp jk_{zm}^{(\nu)} \cdot z} \tag{1}$$

with

$$(f^{(\nu)})' = (x, x, a-x), \quad (p^{(\nu)})' = (a, c, a-d)$$

$$k_{zm}^{(\nu)} = \sqrt{k^2 - \left(\frac{m\pi}{p^{(\nu)}}\right)^2}, \qquad k^2 = \omega^2 \mu \epsilon.$$

$A_m^{(\nu)\pm}$ are the still unknown eigenmode coefficients which are suitably normalized with the factor $T_m^{(\nu)}$ [3], [4], so that the power carried by a given wave is 1 $W$ for a wave-amplitude coefficient of $\sqrt{1W}$.

By matching the tangential field components at the step discontinuity interfaces, the coefficients $A_m^{(\nu)\pm}$ are determined after multiplication with the appropriate orthogonal function. This leads directly to the scattering matrix of each discontinuity. The overall scattering matrix of the total filter section is calculated without introducing transmission matrices by directly combining the single scattering matrices [3], [4], which preserves numerical accuracy.

As has already been introduced for low-loss fin-line filters [6], the computer-aided design is carried out by an optimizing program applying the evolution strategy method [7], where no differentiation step is necessary, which varies the input parameters until a desired value of the insertion loss for a given bandwidth is obtained. An error function $F(\bar{x})$ [6] to be minimized is defined (Fig. 2)

$$F(\bar{x}) = \sum_{i=1}^{N_s} \left(a_{s\min}/a_{21}(f_i)\right)^2 + \sum_{i=1}^{N_p} \left(a_{21}(f_i)/a_{p\max}\right)^2 \stackrel{!}{=} \mathrm{Min} \tag{2}$$